Notes of 2.5 (accelerating convergence)

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Aitken's Δ^2 method
going from linearly convergent to quadratically convergent.

If $p_n \to p$ linearly,
define $\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

proof idea
(all computation)
assume $\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p}$
cross multiply
expand
solve for p
complete the square

Definition (Forward Difference)
Given p_n.
1. define $\Delta p_n = p_{n+1} - p_n$
2. define $\Delta^k p_n = \Delta^{k-1}(\Delta p_n)$ for $k \geq 2$

(by this definition, $\Delta^2 p_n = (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)$)
Aitken can be written neatly as:
$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$

Theorem (rapid convergence of Aitken's method)
If $p_n \to p$ linearly, and $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} < 1$, and $\widehat{p_n}$ is a sequence from the Aitken's method, then
$\lim_{n\to\infty} \frac{\hat{p}_n-p}{p_n-p} = 0$

Steffensen's Method
An algorithm to find fixed points.

idea: with a guess, do standard iterations to get 3 points.
use Aitken's Δ^2 method to get a point using 3 previous points
repeat

Theorem 2.15 (Convergence of Steffensens Method)
Under certain assumptions, Steffensens method converges quadratically.